ON SOLUTIONS

OF MULTI-PARTON 'T HOOFT EQUATIONS

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1 Outline

- An alternative to lattice - diagonalize the Hamiltonian
- On the Light Front - numerics: Light Cone Discretization
- Simplifications (I):

  large N - planar diagrams - single traces

  less dimensions - reductions

  even quantum mechanics (but at N $\rightarrow \infty$)

  supersymmetry
- QCD equations: eigenequations for H_{LC}

  coupled Bethe-Salpeter equations on the LC

  simplifications (II) - Coulomb Approximation
- 't Hooft equations with many partons
- Solutions – numerical
- Solutions – analytical

2 Diagonalizing Hamiltonian

2.1 One way: Light Cone Discretization

$$
\begin{aligned}
P^+ &= \sum_{i=1}^{n} p_i^+, \quad p_i^+ > 0 \\
K &= \sum_{i=1}^{n} k_i, \quad K, k_i - \text{integer} \ \ (> 0),
\end{aligned}
$$

Cutoff $K \Longrightarrow$ partitions $\{k_1, k_2, \ldots\} \Longrightarrow$ states

$$
|\{k\}\rangle = Tr[a^\dagger(k_1)a^\dagger(k_2)...a^\dagger(k_n)]|0\rangle \tag{1}
$$

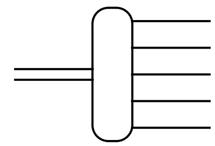
$|\{k\}\rangle \Longrightarrow \langle\{k\}|H|\{k'\}\rangle \Longrightarrow E_n$

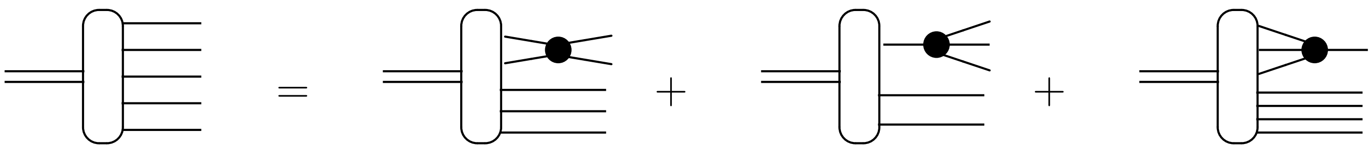
[Brodsky et al.]

2.2 Second way: integral equations in the continuum

- Different cutoff (on parton multiplicity) – directly in the continuum

$$H|\Phi\rangle = M^2|\Phi\rangle \tag{2}$$

$|\Phi\rangle \to \Phi_n(x_1, x_2, \ldots, x_n) \qquad \leftrightarrow$ 



$$M^2\Phi_n(x_1 \ldots x_n) = A \otimes \Phi_n + B \otimes \Phi_{n-2} + C \otimes \Phi_{n+2} \tag{3}$$

- EQUATIONS

$$|\Phi\rangle = \sum_{n=2}^{\infty} \int [dx]\delta(1 - x_1 - x_2 - \ldots x_n)\Phi_n(x_1, x_2, \ldots x_n) Tr[a^\dagger(x_1)a^\dagger(x_2)\ldots a^\dagger(x_n)]|0\rangle$$

EXAMPLE 1: QCD_2 (fundamental fermions)

$$M^2 f(x) = m^2 \left(\frac{1}{x} + \frac{1}{1-x}\right) f(x) + \frac{\lambda}{\pi}\int_0^1 dy \frac{f(x) - f(y)}{(y-x)^2}$$

$$f(x) = \Phi_2(x, 1-x)$$

EXAMPLE 2: SYM_2 restricted to the two-parton sector

There are two coupled equations in the bosonic sector

$$M^2\phi_{bb}(x) = m_b^2\left(\frac{1}{x}+\frac{1}{1-x}\right)\phi_{bb}(x)+\frac{\lambda}{2}\frac{\phi_{bb}(x)}{\sqrt{x(1-x)}}$$

$$-\frac{2\lambda}{\pi}\int_0^1\frac{(x+y)(2-x-y)}{4\sqrt{x(1-x)y(1-y)}}\frac{[\phi_{bb}(y)-\phi_{bb}(x)]}{(y-x)^2}dy+\frac{\lambda}{2\pi}\int_0^1\frac{1}{(y-x)}\frac{\phi_{ff}(y)}{\sqrt{x(1-x)}}dy$$

$$M^2\phi_{ff}(x) = m_f^2\left(\frac{1}{x}+\frac{1}{1-x}\right)\phi_{ff}(x)$$

$$-\frac{2\lambda}{\pi}\int_0^1\frac{[\phi_{ff}(y)-\phi_{ff}(x)]}{(y-x)^2}dy+\frac{\lambda}{2\pi}\int_0^1\frac{1}{(x-y)}\frac{\phi_{bb}(y)}{\sqrt{y(1-y)}}dy$$

and the single one in the fermionic sector

$$M^2\phi_{bf}(x) = \left(\frac{m_b^2}{x}+\frac{m_f^2}{1-x}\right)\phi_{bf}(x)+\frac{2\lambda}{\pi}\frac{\phi_{bf}(x)}{\sqrt{x}+x}$$

$$-\frac{2\lambda}{\pi}\int_0^1\frac{(x+y)}{2\sqrt{xy}}\frac{[\phi_{bf}(y)-\phi_{bf}(x)]}{(y-x)^2}dy-\frac{\lambda}{2\pi}\int_0^1\frac{1}{(1-y-x)}\frac{\phi_{bf}(y)}{\sqrt{xy}}dy \qquad (4)$$

Example 3: YM_2 with addjoint fermionc matter - all parton-number sectors

$$
\begin{aligned}
M^2\phi_n(x_1\dots x_n) &= \frac{m^2}{x_1}\phi_n(x_1\dots x_n) \\
&+ \frac{\lambda}{\pi}\frac{1}{(x_1+x_2)^2}\int_0^{x_1+x_2} dy\phi_n(y, x_1+x_2-y, x_3\dots x_n) \\
&+ \frac{\lambda}{\pi}\int_0^{x_1+x_2}\frac{dy}{(x_1-y)^2}\{\phi_n(x_1, x_2, x_3\dots x_n) \\
&-\phi_n(y, x_1+x_2-y, x_3\dots x_n)\} \\
&+ \frac{\lambda}{\pi}\int_0^{x_1} dy\int_0^{x_1-y} dz\phi_{n+2}(y, z, x_1-y-z, x_2\dots x_n)\left[\frac{1}{(y+z)^2}-\frac{1}{(x_1-y)^2}\right] \\
&+ \frac{\lambda}{\pi}\phi_{n-2}(x_1+x_2+x_3, x_4\dots x_n)\left[\frac{1}{(x_1+x_2)^2}-\frac{1}{(x_1-x_3)^2}\right] \\
&\pm\ cyclic\ \ permutations\ \ of\ \ (x_1\dots x_n)
\end{aligned}
$$

3 This work (JHEP 1106:051, 2011)

- $\mathcal{N} = 1, \quad SYM_4$ on the LC
- Reduce $D = 4 \longrightarrow D = 2 \quad \Longrightarrow \quad QCD_2$ with addjoined matter
- The Coulomb Approximation - keep only most singular (IR) terms in H

1. diagonal in parton multiplicity – can study each p separately, here $p = 2, 3, 4$
2. eigenvalues – spectrum
3. eigenstates – wave functions also in x - space
4. confinement – determine string tension

4 Coulomb divergences

- IR divergences (logarithmic) couple different multiplicity sectors
- Coulomb divergences (linear), but they cancel within one multiplicity
- Can be done independently for each parton multiplicity p

A possibility

- $\longrightarrow$ Solve Coulomb problem first, and then successively add radiation

Simplified Hamiltonian $SYM_4 \quad \Longrightarrow \quad SYM_2 \quad \Longrightarrow H_{Coulmb}$

$$H^{quad}_{Coulomb} = \frac{\lambda}{\pi} \int_0^\infty dk \int_0^k \frac{dq}{q^2} \mathrm{Tr}[A_k^\dagger A_k] \tag{5}$$

$$H^{quartic}_{Coulomb} = -\frac{g^2}{2\pi} \int_0^\infty dp_1 dp_2 \left[\int_0^{p_1} \frac{dq}{q^2} \mathrm{Tr}[A^\dagger_{p_1} B^\dagger_{p_2} B_{p_2+q} A_{p_1-q}] \right. \\ \left. + \int_0^{p_2} \frac{dq}{q^2} \mathrm{Tr}(A^\dagger_{p_2} B^\dagger_{p_1} B_{p_1+q} A_{p_2-q}) \right]$$

5 Two partons

$$|k, K-k\rangle, \quad k = 1, .., K-1 \tag{6}$$

$$\langle k|H|k'\rangle \Rightarrow |\Phi_n\rangle \Rightarrow \Phi_n(k) \overset{FT}{\Rightarrow} \Phi_n(d_{12}) \tag{7}$$

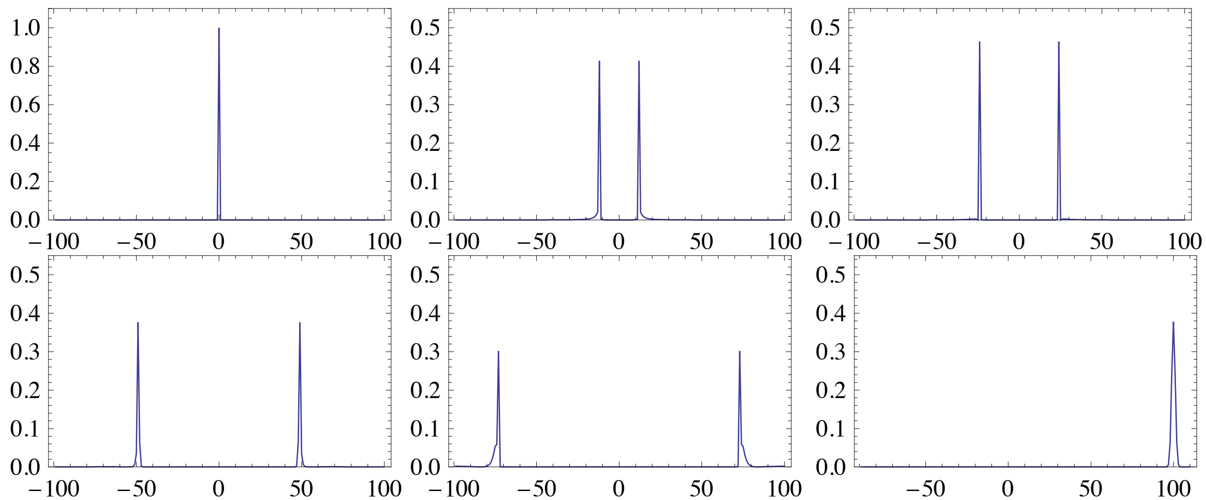


Figure 1: $\rho_n(d_{12}), p = 2, K = 200, n = 1, 25, 50, 100, 150, 199.$

Linear spectrum for two partons

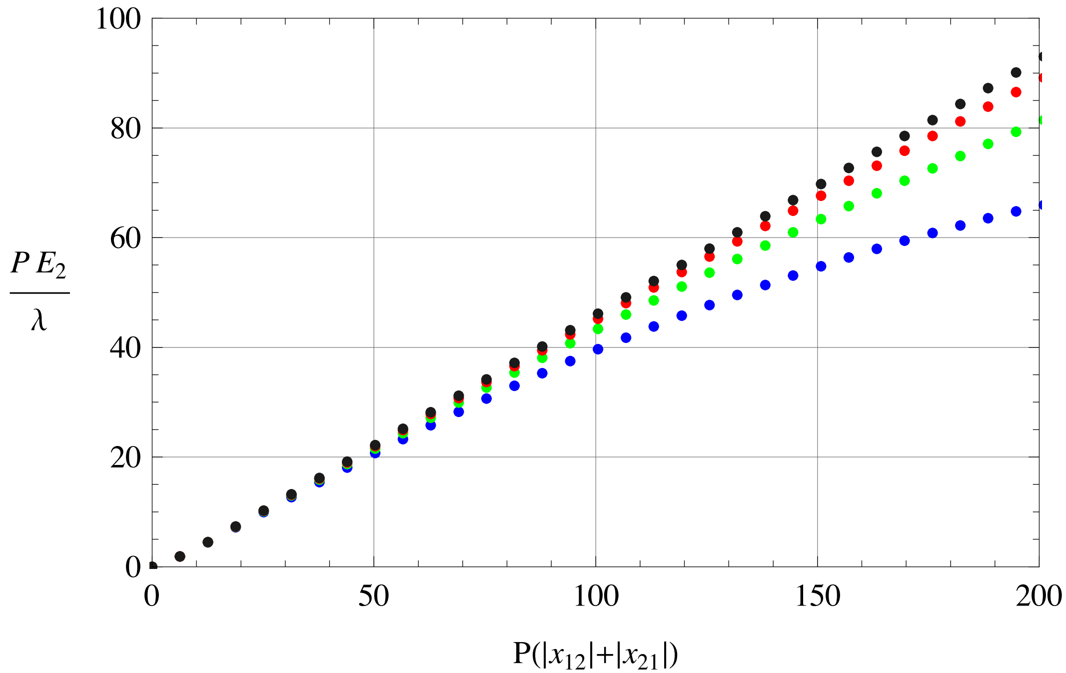


Figure 2: **Eigenenergies of the, p=2, excited states as a function of the relative separation between two partons,** $K = 30, 50, 100, 200$**.**

6 Three partons - generalization of the 't Hooft solution to many bodies

$$|k_1, k_2, K - k_1 - k_2\rangle, \quad k_1 = 1, .., K - 2, \quad k_2 = 1, .., K - k_1 - 1$$

$$\langle k_1, k_2 | H | k_1', k_2' \rangle \Rightarrow |\Phi_n\rangle \Rightarrow \Phi_n(k_1, k_2) \stackrel{FT}{\Rightarrow} \Phi_n(d_{13}, d_{23})$$

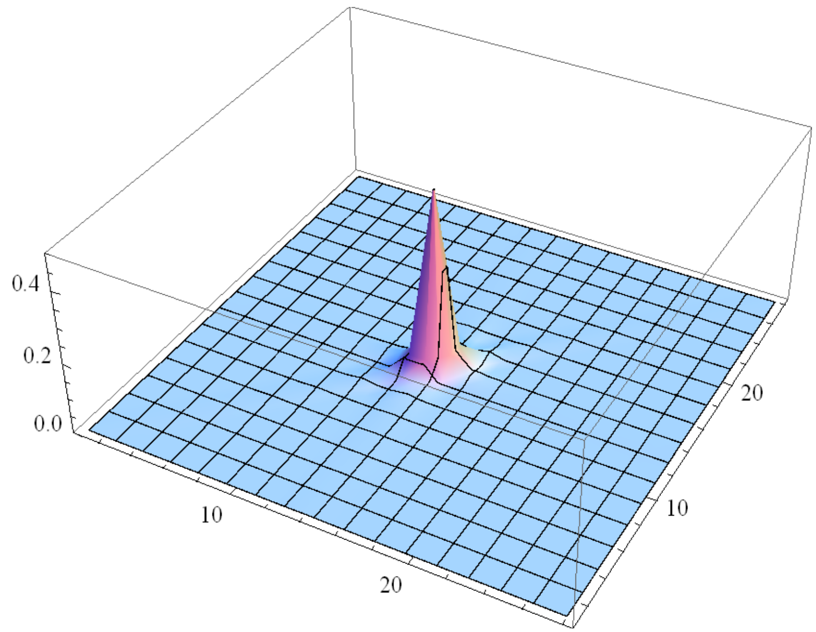


Figure 3: $\rho_1(d_{13}, d_{23})$

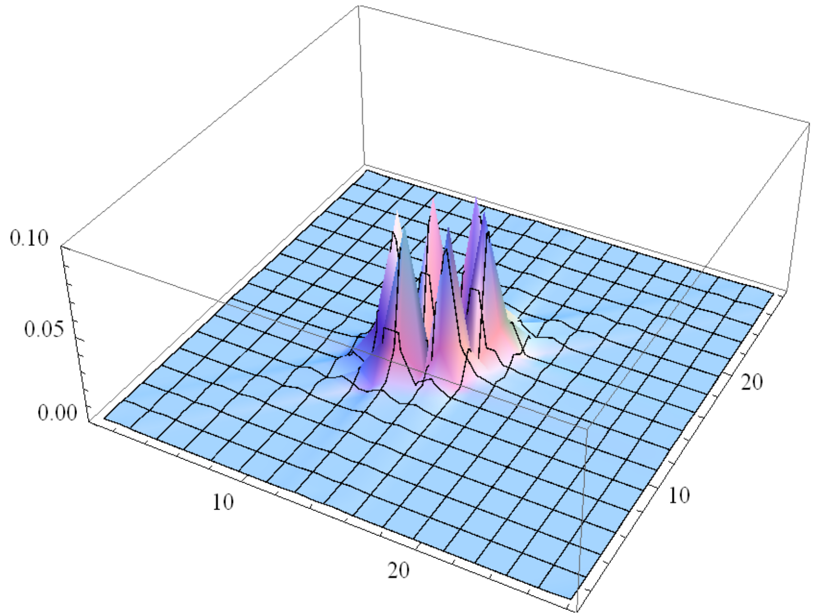


Figure 4: $|\rho_{10}(d_{13}, d_{23})$

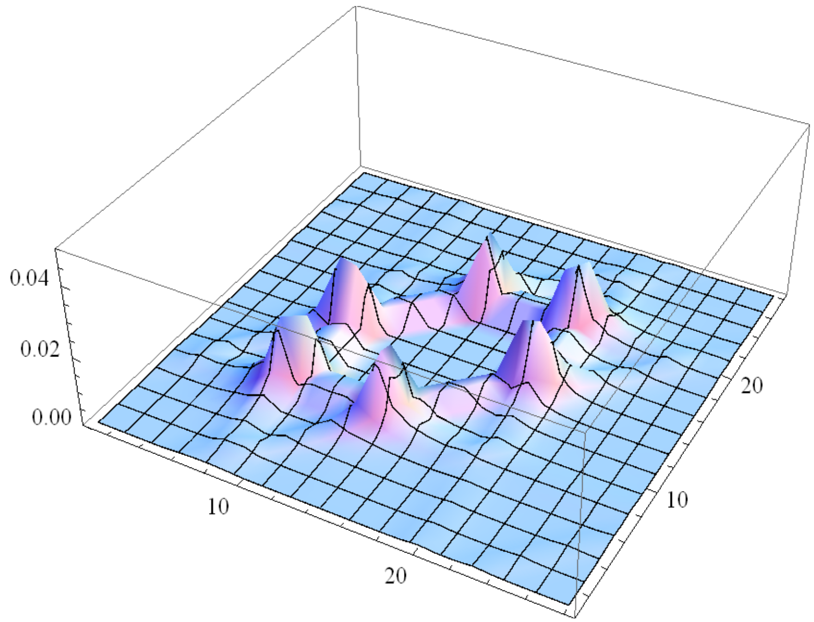


Figure 5: $\rho_{50}(d_{13}, d_{23})$

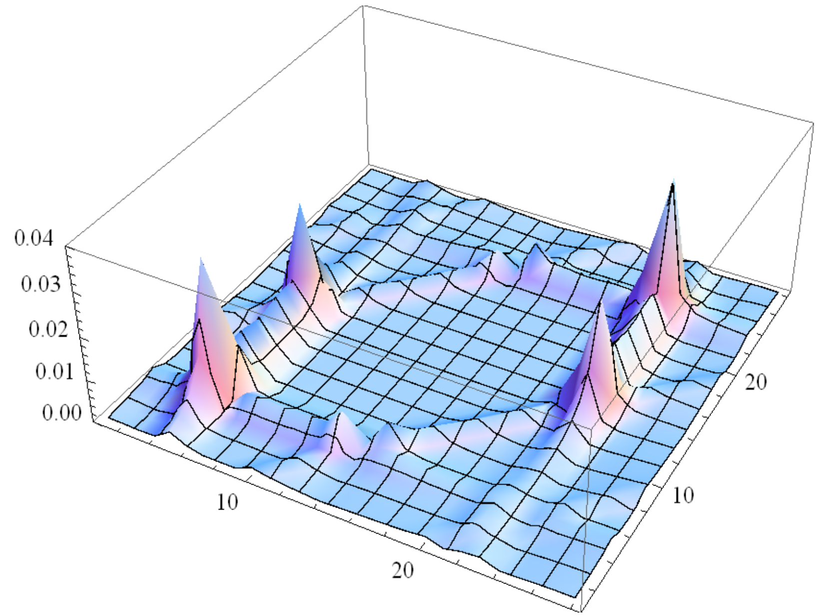


Figure 6: $\rho_{100}(d_{13}, d_{23})$

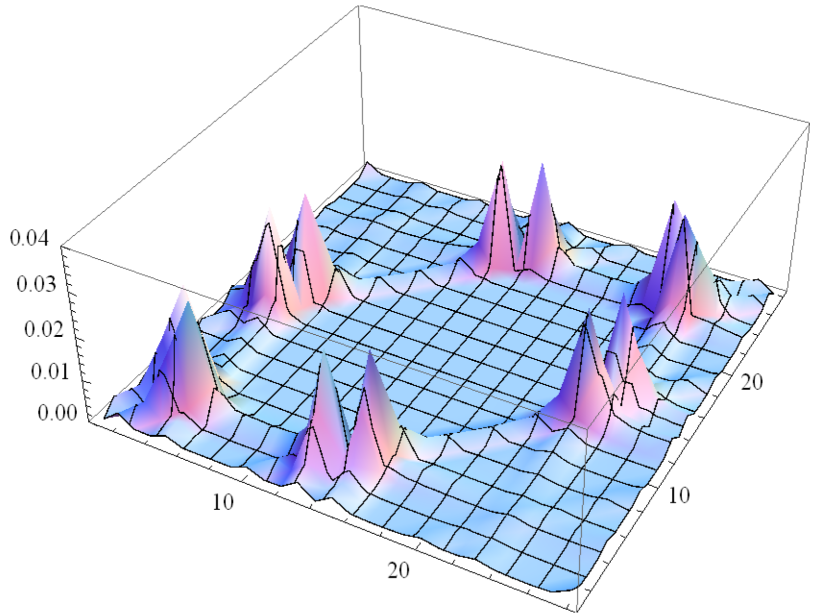


Figure 7: $\rho_{200}(d_{13}, d_{23})$

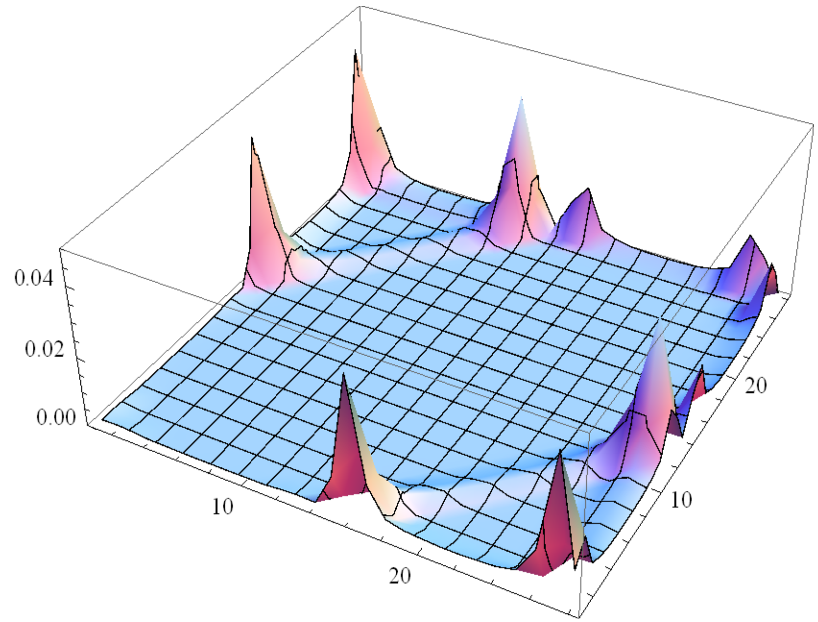


Figure 8: $\rho_{300}(d_{13}, d_{23})$

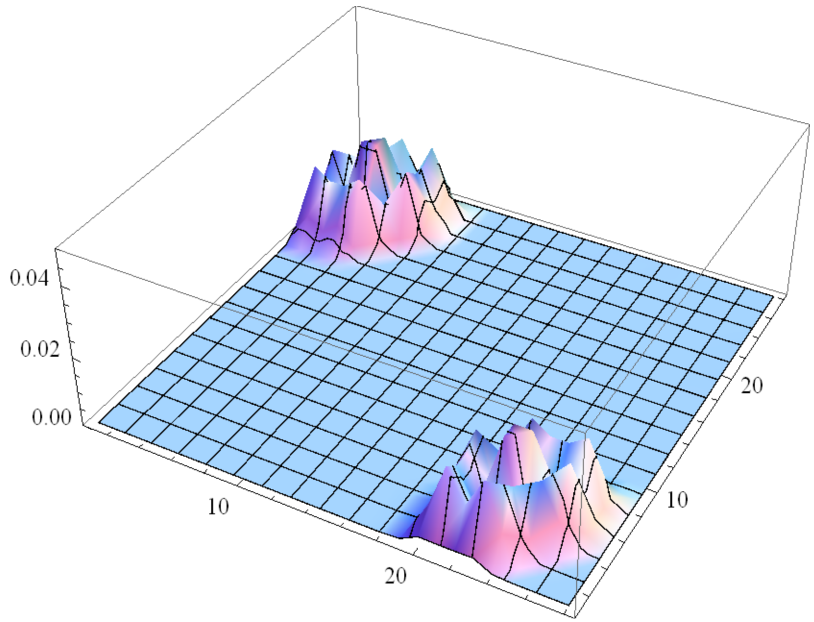


Figure 9: $\rho_{400}(d_{13}, d_{23})$

The highest state

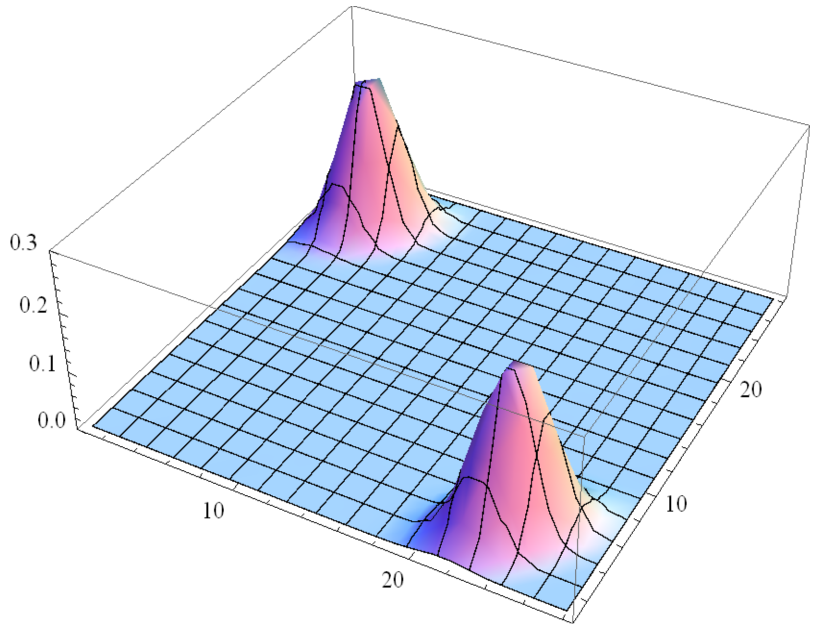


Figure 10: $\rho_{406}(d_{13}, d_{23})$

And on the Dalitz plot

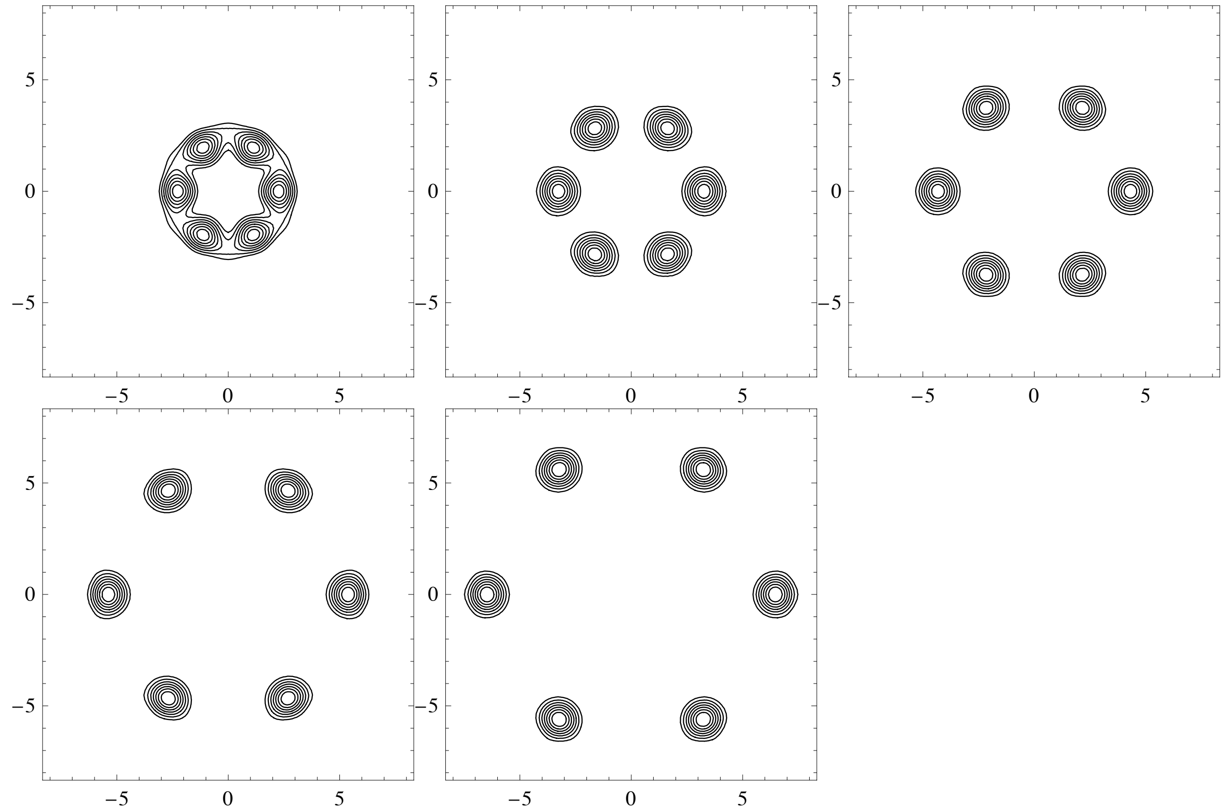


Figure 11: Series B. As above but on the Dalitz plot. Now diquarks are allowed, $d_{min} = 0$

Linear spectrum for three partons

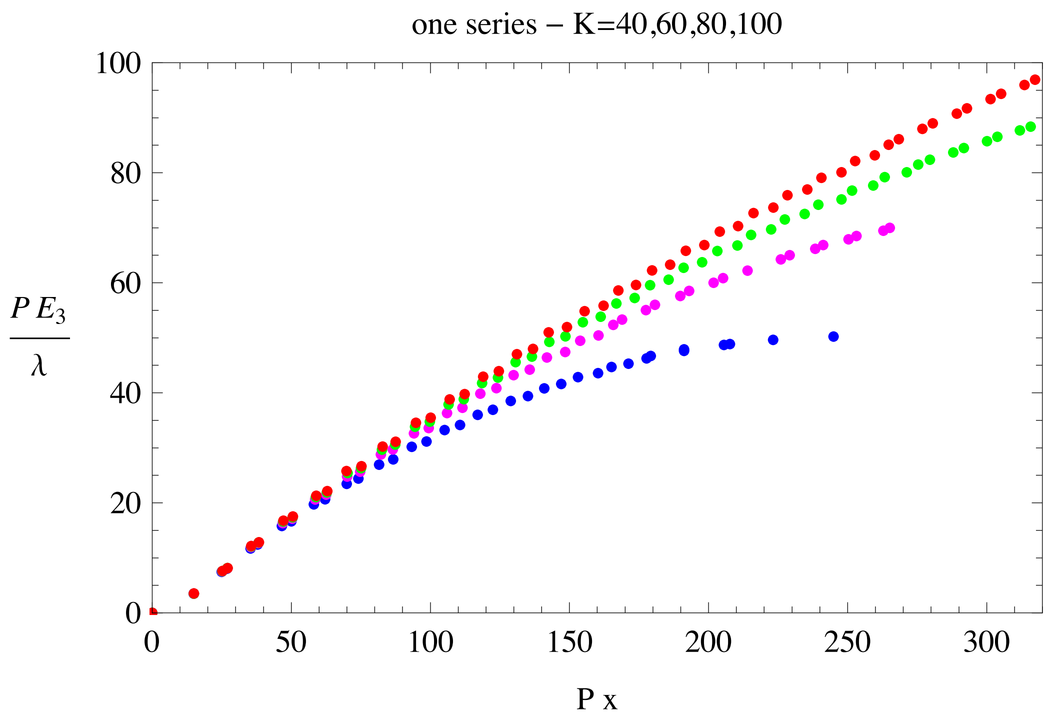


Figure 12: Eigenenergies of the, p=3, excited states as a function of the combined length of strings stretching between three partons.

Four partons

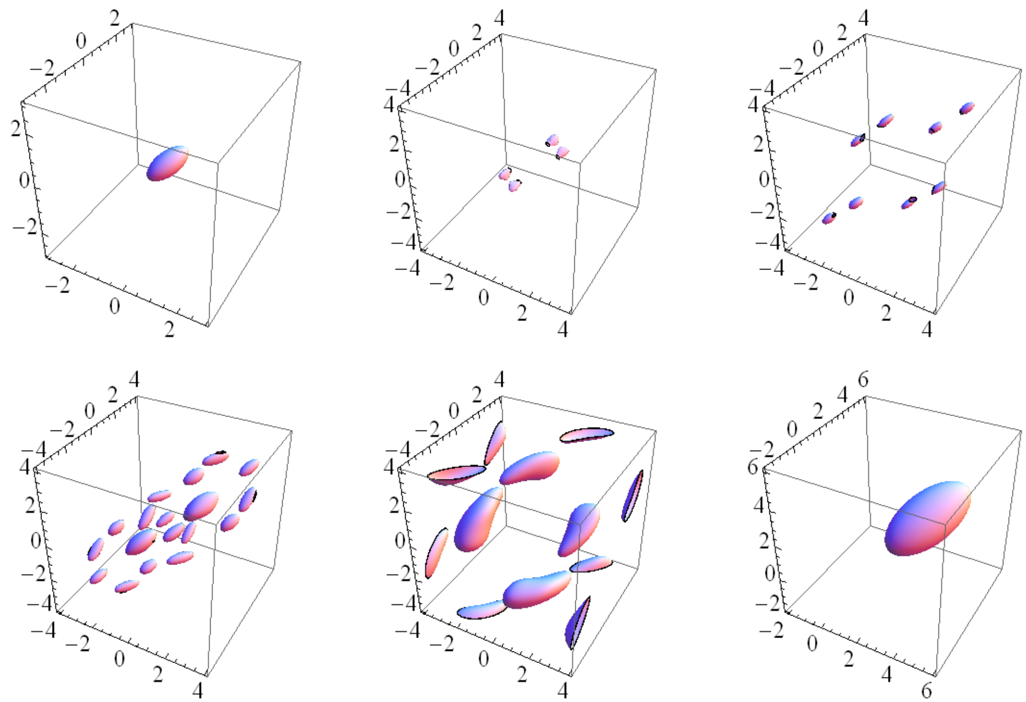


Figure 13: Structure of eigenstates with four partons. Contour plots in three relative distances (d_{14}, d_{24}, d_{34}) for states no. 1,9,35,60,100,165 spanning the whole range of states for $K = 12$, $r_{max} = 165$.

7 Analytic solutions

- Massless quarks

$$\frac{\lambda}{\pi}\int_0^P dk \frac{f(p)-f(k)}{(p-k)^2} = E_C f(p) \longrightarrow Fig.2$$

- Assume that the singularity dominates (e.g. for large E_C) [Kutasov, '95]

$$\begin{aligned}\frac{\lambda}{\pi}\int_{-\infty}^{\infty} dk \frac{f(p)-f(k)}{(p-k)^2} &= E_C f(p) \\ f(k) = \exp{(ik\Delta)} \longrightarrow & E_C = \lambda|\Delta|, \quad \Delta = r_2 - r_1 \end{aligned} \tag{8}$$

- a generic solution - Δ arbitrary
- boundary conditions
- massless quarks $\longrightarrow$ Neumann: $f'(0) = f'(P) = 0$ [Neuberger, '04]

$$\Delta = \frac{n}{2}\frac{2\pi}{P} = \frac{n}{2}a$$

$$f_n(k) = \cos{(\pi n k/P)} = \cos{(\pi n x_F)}$$ ['t Hooft, '74]

Two partons: numerics vs. analytics

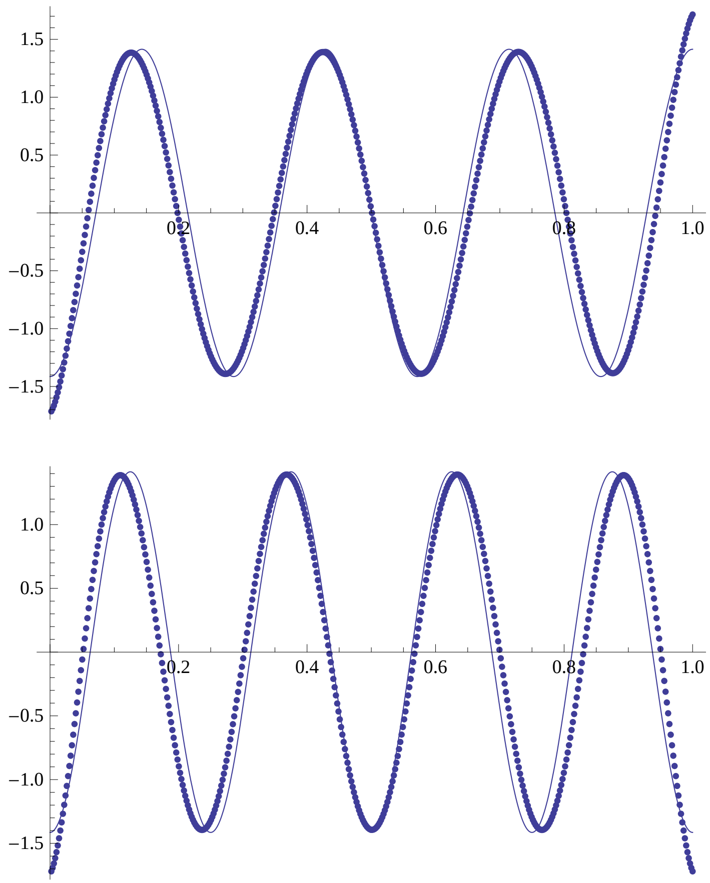


Figure 14: Comparison of numerical (DLCQ) and analytical (WKB) results for the two LC wave functions in the two parton sector

8 Analytic solution in many parton sectors

- Strategy:

  general solution of the asymptotic equation for n partons

  derive boundary conditions (BC) for n partons

  identification of independent (and complete) set of solutions satisfying BC

  classifying solutions w.r.t. their behaviour under Z_n

- n-parton 't Hooft equation

$$\frac{\lambda}{2\pi}\int_0^{p_1+p_2} dk \frac{\psi_n(p_1,p_2,p_3\ldots p_n)-\psi_n(k,p_1+p_2-k,p_3\ldots p_n)}{(p_1-k)^2}$$
$$\pm cyclic\ \ permutations\ \ of\ \ (p_1\ldots p_n)$$
$$= E_C\psi_n(p_1\ldots p_n) \qquad (9)$$

- phase space

$$p_1+p_2+\ldots+p_n = P, \qquad p_i > 0 \qquad (10)$$

only $n-1$ independent momenta,
e.g. for $n=2 \quad \psi_2(p_1, P-p_1) = f(p_1)$

- phase space boundaries: $p_i = 0, \quad i = 1,\ldots,n.$

- Boundary conditions - two partons

$$M^2 f(x) = m^2 \left(\frac{1}{x} + \frac{1}{1-x}\right) f(x) + \frac{\lambda}{\pi} PV \int_0^1 dy \frac{f(x) - f(y)}{(y-x)^2}$$

- $m > 0 \quad \longrightarrow$ Dirichlet
- $m = 0 \quad \longrightarrow$ Neumann
- BC for n massless partons: generalization of Neumann conditions

$$\begin{aligned} p_1 = 0 &: (\partial_2 - 2\partial_1)\psi = 0 \\ p_i = 0 &: (\partial_{i+1} - 2\partial_i + \partial_{i-1})\psi = 0, \quad 2 \le i \le n-2 \\ p_{n-1} = 0 &: (\partial_{n-2} - 2\partial_{n-1})\psi = 0 \\ p_n = 0 &: (\partial_1 + \partial_{n-1})\psi = 0 \end{aligned}$$

[Z. Ambrozinski]

BC follow from a requirement of cancellation of IR divergences at the boundaries of the phase space.

- generic solution of asymptotic $\left(\int_0^{x_i+x_j} \ldots \longrightarrow \int_{-\infty}^{\infty} \ldots\right)$equations in n parton sector

$$\psi(k_1, \ldots, k_n) \;=\; \exp\left(ik_1r_1 + ik_2r_2 + \ldots + ik_nr_n\right) \tag{11}$$

- asymptotic eigenvalue

$$E_C = \frac{\lambda}{2}\Sigma_{i=1}^{n}|\Delta_{i,i+1}|, \quad \Delta_{i,j} = r_i - r_j, \quad n+1 = 1. \tag{12}$$

- How to construct solutions which satisfy BC ??

9 Three partons

- New feature of $n > 2$ sectors: degeneracy $\longrightarrow$ use more trial functions with the same eigenvalue

Sufficient set for $n = 3$

$$
\begin{aligned}
\Psi_1 &= \exp\left(+i(k_1 r_1 + k_2 r_2 + k_3 r_3)\right) \\
\Psi_2 &= \exp\left(-i(k_1 r_1 + k_3 r_2 + k_2 r_3)\right) \exp\left(i2Pr_1\right) \\
\Psi_3 &= \exp\left(+i(k_2 r_1 + k_3 r_2 + k_1 r_3)\right) \\
\Psi_4 &= \exp\left(-i(k_3 r_1 + k_2 r_2 + k_1 r_3)\right) \exp\left(i2Pr_2\right) \\
\Psi_5 &= \exp\left(+i(k_3 r_1 + k_1 r_2 + k_2 r_3)\right) \\
\Psi_6 &= \exp\left(-i(k_2 r_1 + k_1 r_2 + k_3 r_3)\right) \exp\left(i2Pr_3\right)
\end{aligned}
$$

Or in terms of independent momenta and coordinate differences

$$
\begin{aligned}
\psi_1 &= exp\left(i(k_1\Delta_{13} + k_2\Delta_{23})\right) \exp\left(iPr_3\right) \\
\psi_2 &= exp\left(i(k_1\Delta_{21} + k_2\Delta_{23})\right) \exp\left(iP(r_3 + \Delta_{13} + \Delta_{12})\right) \\
\psi_3 &= exp\left(i(k_1\Delta_{32} + k_2\Delta_{12})\right) \exp\left(iP(r_3 + \Delta_{23})\right) \\
\psi_4 &= exp\left(i(k_1\Delta_{13} + k_2\Delta_{12})\right) \exp\left(iP(r_3 + \Delta_{23} + \Delta_{21})\right) \\
\psi_5 &= exp\left(i(k_1\Delta_{21} + k_2\Delta_{31})\right) \exp\left(iP(r_3 + \Delta_{13})\right) \\
\psi_6 &= exp\left(i(k_1\Delta_{32} + k_2\Delta_{31})\right) \exp\left(iPr_3\right)
\end{aligned}
$$

- Necessary condition for BC: on each plane some subsets have to have the same dependence on all other (not fixed) variables.

  E.g. on $k_1 = 0$ boundary cancellations may occur only within (1,2) , (3,4) and (5,6) pairs.

- Indeed, for integer (in units of $2\pi/P$) Δ's, all BC's are satisfied by

$$\psi_{r,s}(k_1, k_2) = \Sigma_{i=1}^{6}\psi_i = \psi^{singlet}, \qquad \Delta_{13} = \frac{r}{2}, \ \Delta_{23} = \frac{s}{2}, \ \ r, s \ \text{ even}$$

- Z_3 covariant solutions can be constructed as well

$$\psi_{r,s,\nu}(k_1, k_2) = \psi_1 + \lambda\psi_5 + \lambda^2\psi_3 + \psi_2 + \psi_4 + \psi_6$$

$$\Delta_{13} = \frac{r+\nu}{2}; \quad \Delta_{2,3} = \frac{s-\nu}{2} \quad \nu = \pm\frac{1}{3}, \quad \lambda = e^{2\pi i\nu}, \quad r, s \quad odd.$$

  this quantization follows from

$$\exp\left(iP\Delta_{13}\right) = \lambda^2, \quad \exp\left(iP\Delta_{23}\right) = \lambda,$$

  which generalizes the $\exp\left(iP\Delta_{12}\right) = \pm 1$ from the two parton case.

- all pairs (r, s) generate overcomplete sets
- for a complete basis it suffices to use

$(r, s) = (2n, 2l)$ and/or $(2l, 2n)$, $\quad 0 \leq l \leq [n/2]$.
for each eigenvalue $E_C = \frac{\lambda}{2} La$ and $\nu = 0$,
where the "combined length of strings" $L = 2n$.
$\longrightarrow$ each $E_C(n)$ has degeneracy

$$g_n = \begin{cases} n+1, & n \quad even \\ n, & n \quad odd \end{cases} \tag{13}$$

and for $\nu = 1/3$:

$$L^I = 2n + 1 + \nu, \qquad L^{II} = 2n + 3 - \nu, \tag{14}$$

$$(r, s)^I = (2n + 1, 2l + 1), \qquad (r, s)^{II} = (2l + 1, 2n + 3) \tag{15}$$

9.1 Comparison with numerical results

- Profiles of non degenerate states agree very well, c.f. Table 1 for $\nu = 1/3$
- Eigenenergies differ by 50% for the lowest state.

  The discrepancy goes down to 30% around $no = 13 \leftrightarrow$ WKB.

$num. - no's$	$anal. - (r,s)$	$\| < num\|anal > \|^2$	$LP/2\pi$	E_{anal}	E_{num}
1	(0,0)	1.0	0	0	0
4	(2,2)	.96	2	39.5	22.0
(2,3)	(1,1)	.96	4/3	26.3	11.3
(5,6)	(1,3)	.93	8/3	52.6	29.3
(7,8)	(3,3)	.91	10/3	65.8	39.0
(12,13)	(3,5)	.87	14/3	92.1	58.2

Table 1: First six states in the $\nu = 0, 1/3$ sector, comparison with numerical (DLCQ) calculations.

- for higher states (i.e. with degeneracy): analytical solutions with degeneracy g correspond uniquely to a group of g numerical eigenstates (substantial overlaps)

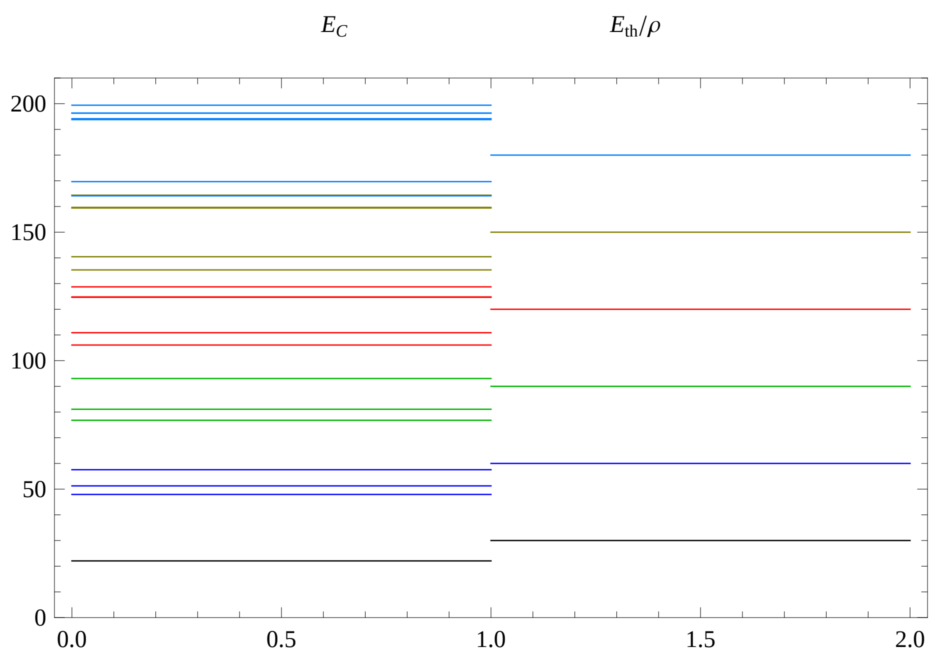


Figure 15: Correspondence between the numerical (left) and analytical (right) spectra. Only Z_3 singles are shown. Analytic levels are g-fold degenerate, here g=1,3,3,5,5 and 7 respectively. $\rho = 1.3$

- High eigenvalues - can test completeness and WKB by comparing the entropy, or rather the number of states with energy below E.

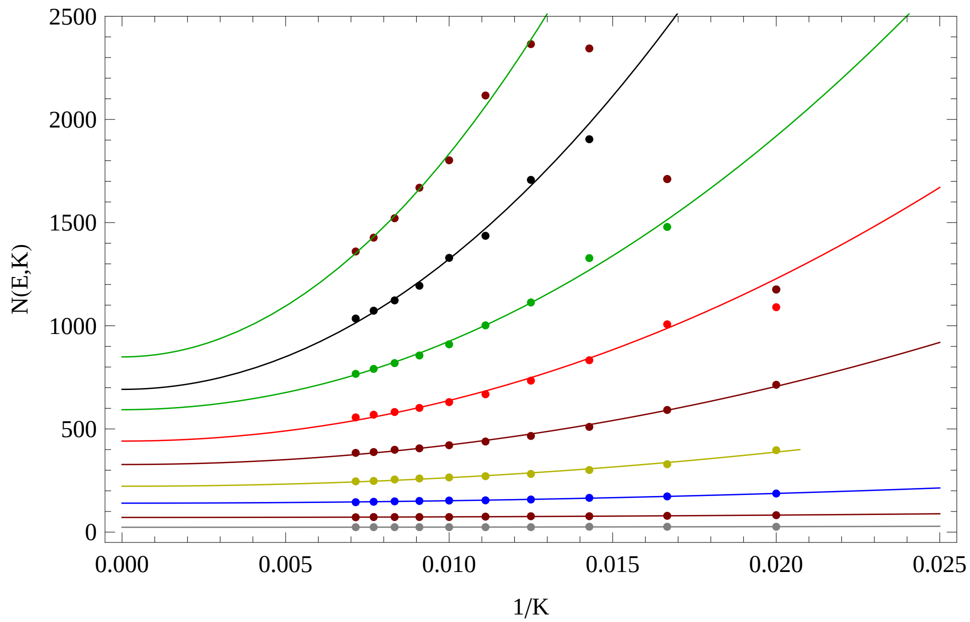


Figure 16: Energy distribuant $N(E, 1/K)$ and its extrapolation to $K = \infty$

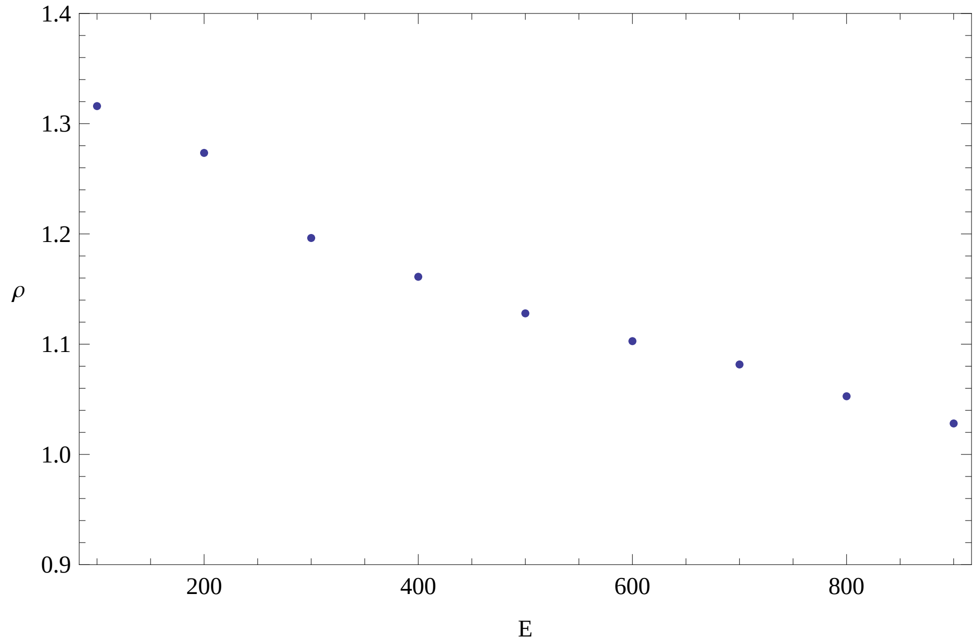


Figure 17: Effective scale factor obtained from $N_{num}(E, K = \infty) = N_{anal}(E/\rho)$

10 Four partons

- Trial states are direct generalization of symmetric sums from the $n = 3$ case.
- They are characterized by a triple of integers (d_{12}, d_{23}, d_{34}), $d = \Delta P/2\pi$.
- They DO NOT satisfy our boundary conditions !
- However their simple combinations DO .

Procedure

1. Generate all sets of above triples which satisfy
$$\Sigma_i^4 |d_{i,i+1}| = L = 2n, \tag{16}$$
for a given n.
2. Identify linearly independent subset of corresponding trials
3. Search for the linearly dependent combinations on the boundary planes by inspecting generalized Wronskians of corresponding partial derivatives.
4. Identify combinations satisfying our boundary conditions.
5. Organize states found in pt. 4 by choosing some labeling scheme.
6. Check completness of this basis as in the three parton case.

Results

A. Indeed a series of simple linear combinations, which satisfy boundary conditions (BC) on all boundary planes, exists.

B. Only combinations, which appear, contain one (singles), two (doubles) and three (triples) basis functions from step (2).

C. Each independent trial function from step (2) appears once and only once in one of the combinations. All independent trials are used.

D. Relative coefficients of all combinations found are very simple: all 1's in triples, and 1 and 2 in doubles. This finds a nice explanation upon the detailed inspection below.

E. All combinations are orthogonal even though the original basis, found in 2, was not.

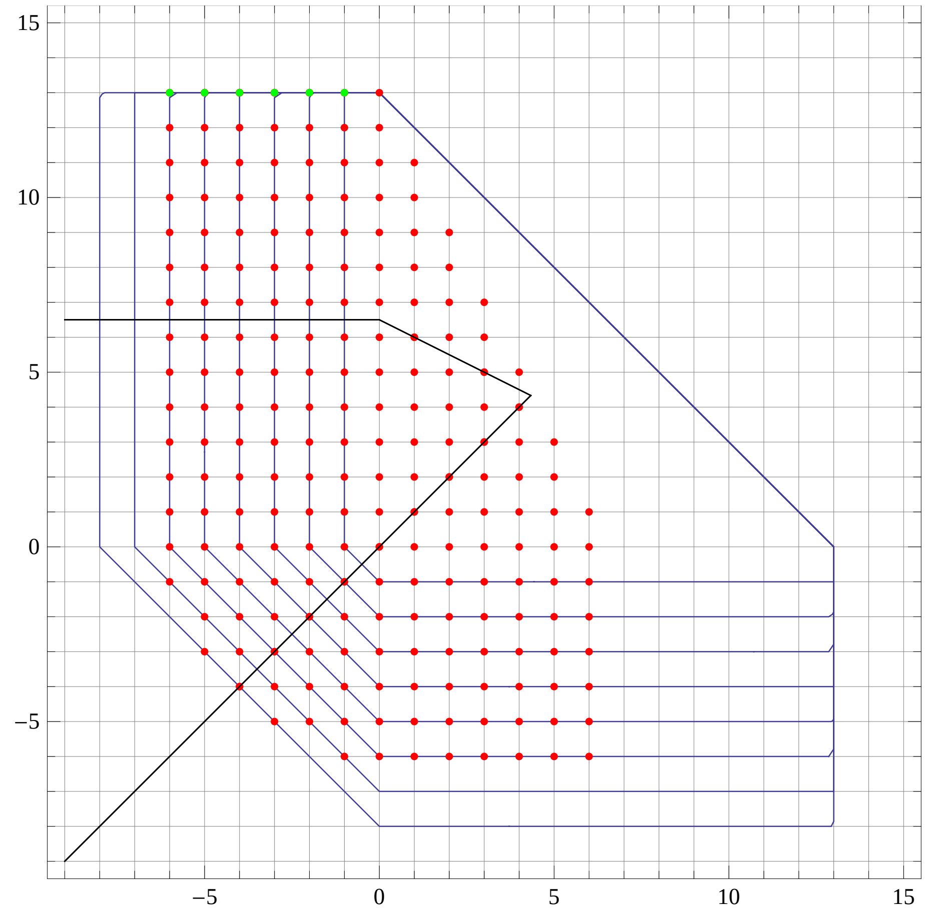
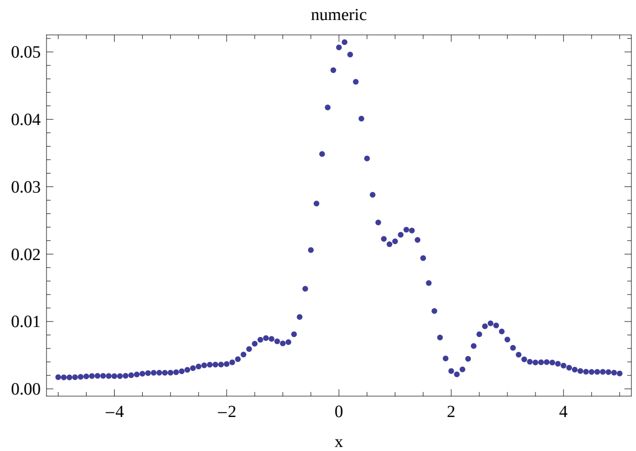


Figure 18: Solutions with 4 partons on the $(d_{12}, d_{23}) = (i, j)$ plane, together with the contour plots (blue) of $|d_{12}| + |d_{23}| + |a - d_{12} - d_{23}| = 2n - |a|$ for fixed $a = d_{12} + d_{23} + d_{34} = n, n-1, n-2, ...$; n=13. Reflections across the black lines provide triples which satisfy BC.



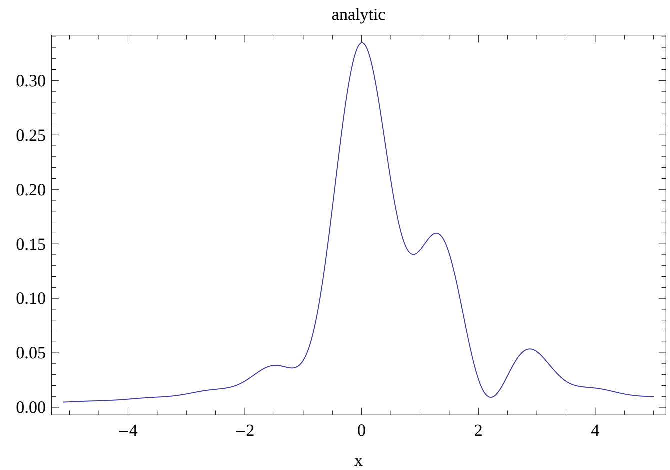


Figure 19: x profile: numeric (left) and analytic (right), $y = z = 1.3$

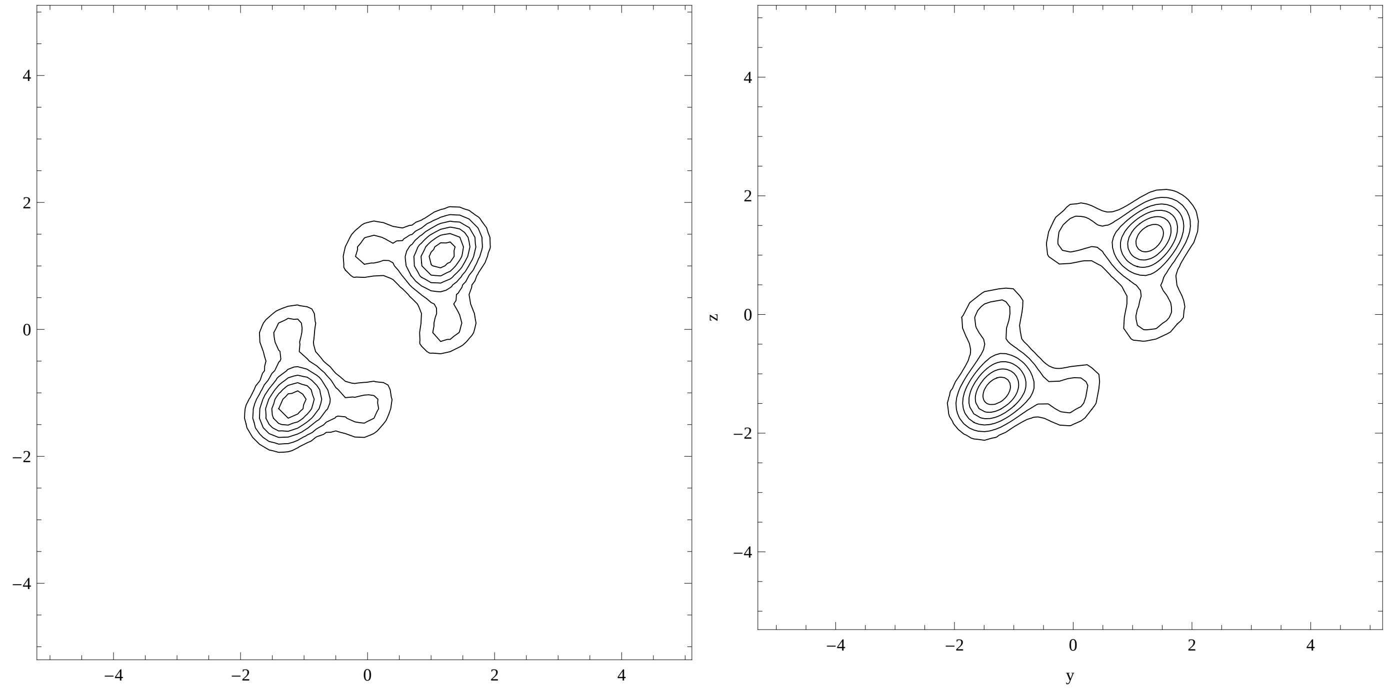


Figure 20: (y, z) contour plots of the same profile: numeric vs. analytic as above, $x = 0$

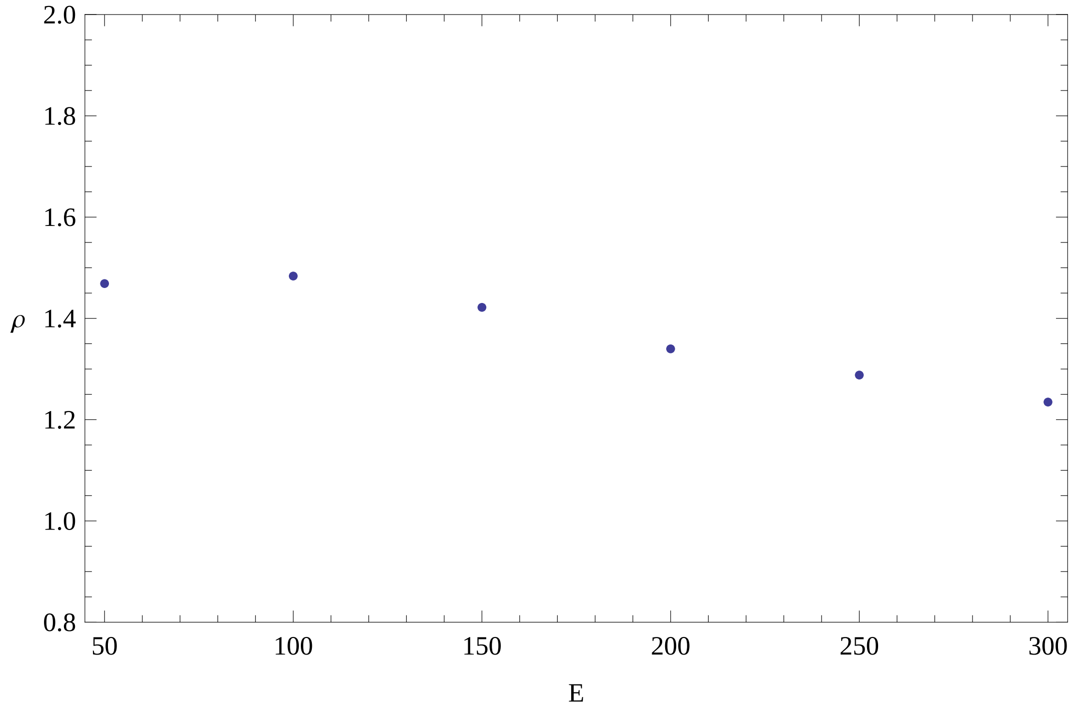


Figure 21: Scale factor for four partons

11 Arbitrary number of partons p

$p = 5$ - similar to p=4: trials, basis of independent solutions,
Wronskians $\Rightarrow$ combinations which satisfy BC (more than triples: 4-,6-,12- plets)

$\Longrightarrow$Rules (emegred from analyzing p=4,5)

Rule I (to generate basis of trial solutions)

- generate all closed loops (made of p "bits") with size d and energy L
- mod out Z_p and IZ_p
- sum over d at fixed L

Rule II (to construct combinations satisfying BC)

- Solutions with the same values of $\{d's\}$ form combinations which satisfy BC's.
  e.g. $(1, 0, 2, -3)$ and $(0, 1, 2, -3)$ for $p = 4$

Counting states (for $p \leq 6$)

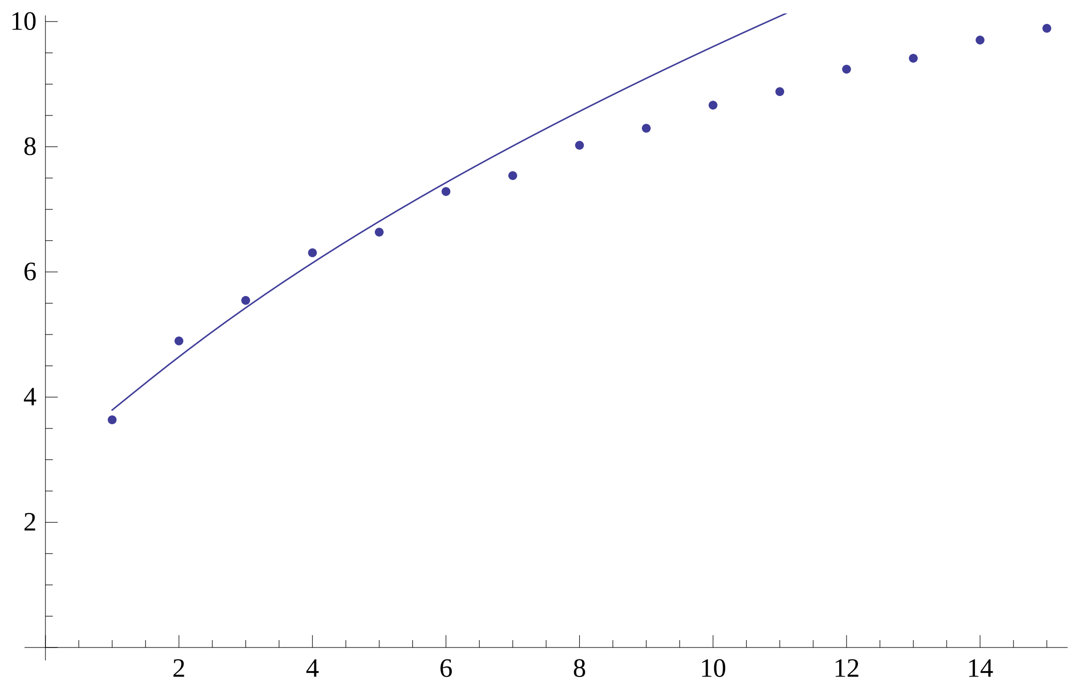


Figure 22: Entropy of solutions (vs. M^2/λ) from the first six multiplicity sectors.

$$\rho(M) \sim \exp M/T_H, \quad T_H = \frac{1.6 - 1.7}{\sqrt{\pi}}\sqrt{\lambda} \leftrightarrow (1.3 - 1.4) \;\; [Bhanot,\; et.al \;\;]$$

12 Summary

- Need a string-like counting of states for arbitrary $p > 4$
- Interpretation of T_H - confirmation with higher p ?
- Green's functions $\longrightarrow$ solve the hierarchy by Gauss elimination !
- Add transverse degrees of freedom ??

EU grant (via Foundation for Polish Science)

Jagellonian University International PhD Studies on Physics of Complex Systems

- 1 M Euro
- 4 years
- 14 PhD students (1/2 - 2 years abroad)
- 17 Foreign Partners: J. Ambjorn, H. Nicolai, S. Sharpe, J.P. Blaizot ...
- 8 Local Supervisors